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# The teaching of Mathematics and the historical development of Euclidean space

# La enseñanza de las Matemáticas y el desarrollo histórico del espacio Euclidiano

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## Abstract

The historical analysis of a concept allows us to understand its evolution and determine the processes of construction of that knowledge, in the case of the Euclidean space, the theoretical representation entails understanding difficulties during its teaching, which can be attributed to the epistemological obstacle as it will be examined in this article. The historical development of the concept of space is briefly analyzed, in documentary form, first within geometry and later in its generalization as a vector space. Finally, the historical perspective is contrasted, to succinctly examine a didactic proposal, which establishes the conclusion that it is necessary to build knowledge based on the elements that characterize epistemic development, which will possibly originate new pedagogical proposals.

**Space, Geometry, Didactics**

## Resumen

El análisis histórico de un concepto permite comprender su evolución y determinar los procesos de construcción de ese conocimiento, en el caso del espacio Euclidiano, la representación teórica conlleva dificultades de comprensión durante su enseñanza, que puede atribuirse al obstáculo epistemológico tal como se propone en este trabajo. Se analiza brevemente, en forma documental el desarrollo histórico del concepto de espacio, primero dentro de la geometría y después en la generalización como espacio vectorial. Finalmente se contrasta la perspectiva histórica, para examinar sucintamente una propuesta didáctica, que establece la conclusión de que es necesario construir el conocimiento en base a los elementos que caracterizan al desarrollo epistémico, lo que podría originar nuevas propuestas pedagógicas.

**Espacio, Geometría, Didáctica**

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## Introduction

It is to be considered that there are several stages of mathematical thought in relation to its formal construction process. Under this perspective we find the historical element of Pythagoreanism as the starting point of mathematics and its philosophical reflection of epistemological character. It can be continued in the same way, with Euclid and the birth of the axiomatic method for the construction of mathematical knowledge.

In the same context is Archimedes, as the person who applies with technical mastery the processes of approximation and mechanical solutions to solve mathematical problems in an ingenious way. In the Middle Ages we find Bacon, a philosopher who considers that the philosophical vein of mathematics is found in the construction of the formal knowledge of science.

Continuing with the historical development of mathematics, a fundamental turn originates with Cartesianism, whose thought is installed as a toral knowledge for the construction of mathematical elements under the process of analysis, as the tool to develop new concepts and solve mathematical problems.

According to Arancibia (2016, p 10):

*"The Cartesian notion of space corresponds to a specific concept situated within a particular conception of the world. Although it can be applied to describe and understand material objects, it does not give way to a defined way of projecting or constructing space-that is why the description of works of art, more radical in their ways of projecting space than architecture, is more productive. This is not only because of the descriptive and metaphysical nature of the concept, but probably because for Descartes, the material world is a given fact and created by God, and therefore does not give off a projective intentionality, but his aim is to try to establish a firm truth in science in relation to the existing world around him."*

In the search for a firm truth Descartes developed the philosophical idea of rationalism, where the main element of knowledge is reason, establishing with this philosophical stance, one of the bases of human knowledge.

It is considered that the pre-relativistic space-time relationship, according to Penrose (2019), corresponds to a representation in terms of a geometric framework in the motion of particles and bodies, using as a basis the Aristotelian, Galilean and Newtonian structures for space, in which the reference to Euclidean space is always found.

Later in the historical timeline, the discussion of the concepts of space and time is sketched by the speculations of Kant's critical-mathematical philosophy in the form of *a priori* knowledge, in order to lead to a methodological resource that is applicable both to the natural sciences and in general.

Since the systematization of geometry by Euclid, it was considered the science of space; in the 19th century, the methodology applied began to be questioned, since it had two variants, the analytical and the synthetic, the debate on these methods significantly affected the teaching, the rigor of mathematical work and its progress.

The historical development of algebra in the 19th century led to the appearance of algebraic geometry, which renewed the methods of the study of curves and surfaces based on synthetic and analytical geometry, and with the appearance of projective geometry, non-Euclidean geometry was born.

The above brought as a consequence the appearance of vector space as a concept to develop hyperspace, which currently has repercussions in the teaching process, in difficulties of explanation and understanding, being this even more complicated than for Euclidean space, which must be demolished by didactic proposals implemented in various pedagogical alternatives of educational mathematics.

Thus it is possible to ask whether, first, the logical structure of a concept is sufficient to articulate learning and, second, whether and with which elements historical-conceptual reconstruction can help to develop didactic strategies. We propose to discuss these two questions.

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On the other hand, in the book Geometría (Bruño, 1970), in the preliminary definitions it is stated that: "The observation of material objects provides us with the ideas of space, volume, surface, line and point. A material body occupies a portion of absolute space", and also considers that: "once these notions are acquired, one can dispense with their experimental origin and conceive of a volume, a surface, a line, a point, independent of any material body".

Thus, taking up again what Pecharroman (2013) explained, the didactic process of geometry has as an initial obstacle the discordance of reality with the ideal representation of the object, being that, of the point, the line and the plane there are only ideal representations, of the real objects, they do not exist, that is, the concrete reality is evoked by means of an object, in an abstract form.

It can be considered that space is an epistemological obstacle, due to the exclusion of its conceptual roots in Cartesian rationalism, which considers that the whole is made up of parts, so this conception is the point to develop the conceptual formation of vector space and therefore the generalization of space in mathematical form.

To systematize the process of conceptual approach to the idea of space, there are proposals such as those of Piaget, called of spatial concepts and that of Pierre van Hiele and Dina van Diele-Geldof that establish five hierarchical levels to describe the understanding and mastery of spatial notions and skills (Arancibia, 2016).

Thus, according to Gómez-Carrasco, et al. (2019), the historical reconstruction of space shows that the use of language is very important for the identification and classification of objects surrounding the environment and to symbolize their representation; however, logic is subsequently resorted to in order to establish the intrinsic characteristics of the figures.

The observation of the development of these ideas, allows us to recognize that the inquiry of a perspective for the didactic understanding and transfer of concepts should be approached, for its investigation, based on a phenomenological perspective.

**Methodology**

The theoretical character of science and history, influence the methodology in a reciprocal way and in relation to mathematics; because its structure is the basis of intrinsic logic, which participates in the construction of mathematical knowledge and by the part of history, which reproduces the image of the world and of man, this finally has repercussions in a philosophy, known as the theory of unique conceptions.

In the formalization of knowledge, one passes from concrete elements to abstract concepts by means of an idealization of reality, having the truth of knowledge as a guiding thread and knowledge as the ultimate goal. Thus, it can be considered that the development of predicative logic occurred when it turned to the fundamental element of knowledge, which is the attainment and demonstration of truth.

However, according to, Descartes (2015), considers that the very serious defect of Aristotle's logic, is in its inability to invent, since it is considered that: "The syllogism cannot be a method of discovery, since the premises (which may be false) must necessarily lead to the conclusion".

Under this perspective, one must "Put culture in a state of permanent mobilization, replace closed and static knowledge, by an open and dynamic knowledge, dialecticize all experimental variables, finally give reason reasons to evolve" (Bachelard, 2018), and thereby open the opportunity for inquiry of conceptual construction.

Also language as an external element, is seen in terms of formalization as a vehicle to transmit knowledge, in the form of a truth that is implicit in it and that has the fundamental characteristic of being objective, i.e. the relationship that keeps the value it contains, is the product of a predicative representation. Under this vision, the development of the concept of space is analyzed in this essay.

Thus in current geometry, only shape and size are considered, so that geometric solids have three characteristics, length, width and height; surfaces are the limits that separate the solid or body from the space that surrounds it, with its fundamental characteristics of length and width.

The main obstacle to construct mathematical space is the Archimedean property, which established that two magnitudes could not be compared unless a multiple of one of them could exceed the other, so the sum of areas with lengths or with volumes could not be developed, Descartes' innovative proposal was to consider that the product of two lines is a line, not an area.

This allowed to interpret all geometric magnitudes in linear form, (Rodriguez, 2022) using algebraic operations and to perform operations with segments of straight lines, that fulfilled the condition of a given relationship, which graphically is possible to represent with two segments, one of length X as axis and another of length Y also as axis, separated by an angle between the two, this idea was imposed by necessity, because originally Descartes used an axis without reference to the other. Descartes also proposes the philosophy of rationalism based on the ideas of Aristotle, and in terms of proceeding by operations that depend only on reason, independently of experience, as is the case of mathematics being considered as a rational science.

There are different shades of this position, such as Kant's critical rationalism in which the a priori of reason correspond to the experience that predefines and organizes them. Hegel, for his part, considers that rational thought is capable of reaching absolute truth insofar as its laws are those that obey a real representation.

Currently rationalism considers that reason is in the possibility of reaching the real, through scientifically elaborated knowledge, taking into account the historical evolution of reason, of an open or dialectical rationalism (Bachelard, 2018).

Considering Cartesian rationalism, and as a fundamental reference to the work of Descartes called Discourse of Method, in the second rule, it is established that: "divide each of the difficulties that I examine in as many parts as possible and in as many as their best solution requires" (Descartes, 2015).

This rule establishes what etymologically is: *analuein*, that is to say, to untie, so that corresponds to the translation, it defines the concept of analysis as the procedure of decomposing a defined whole in its parts or components, as the analysis of a concept.

In mathematics, the concept of analysis has two meanings, one of Greek origin that recognizes analysis as the method of demonstration that starts from a conclusion, and then inversely arrives at the synthesis. The second, modern version, corresponds to the name applied to the branch of mathematics that studies the dependence relationships of certain magnitudes.

Under these ideas, a didactic strategy can be established as a perspective to investigate the development of some concepts. For all of the above, the methodological process to develop the inquiry begins with a staging, starting from the Euclidean ideas and their historical transit, in terms of epistemic obstacle and the *analuein*, referred to a course of linear algebra.

The construction of the mathematical space of the linear algebra course was based on emphasizing two fundamental operations, addition and multiplication, following the basic idea of Cartesianism that the whole is equal to the sum of the parts in the construction of dependence, independence, base and dimension, following a path to avoid the epistemic obstacle from the historical perspective.

Taking into account human nature, which is defined in terms of a subjective voluntarism of learning, it was possible to consider a research methodology that is related to a qualitative and ideographic vision of science, which is characterized by the interpretation of phenomena as they are experienced, lived and perceived by the student, eliminating the description, and taking into account the scientifically valid and useful essence of the didactic phenomenon. Due to the above, being this an exploratory, descriptive and totally documentary study, and having as a basis to analyze a perspective to develop a didactic strategy, an interview was organized on the subject of learning vector spaces and their linear transformations, under a subjective inquiry structure.

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Due to the prevailing Cencia et al. (2021) situation of the Covid19 pandemic of virtual work, a random sample was not selected by any statistical method, but rather by the accessibility of each student who freely decided to answer, the student population on which it was applied was formed by groups of teachers who agreed to participate.

As a preliminary analysis, an unstructured interview was conducted with 8 students, out of a group of 10, who freely decided to participate, on the topics of vector spaces and linear transformations, under a methodology of subjective inquiry, and based on an ontological revision of nominalism, which is defined by the epistemic vision of anti-positivism, as outlined in this essay, and also considering the epistemic in its historical character.

This is developed through the rationalism applied by Bachelard (2018), which allows developing a historical analysis of the formation of science to establish the difference between each scientific knowledge, and concludes that there is no general science, but specific sciences and they are characterized by having different problems to reach the scientific truth.

Furthermore, the process of advancement of the sciences and their development are not made in a continuous and linear way, but by modifications or radical changes in their categories of knowledge and methodologies, which at the time had been considered as fundamental, i.e. as dogmas or traditions.

However, modifications or radical changes are known as epistemological breaks, which are obtained by the characterization of science in its historical development, in its phases or steps, from one phase to another or from one step to another, allowing to distinguish scientific theories from non-scientific discourses according to, Bachelard (2018, p 45).

*When one investigates the psychological conditions of the progress of science, one soon comes to the conviction that one must pose the problem of scientific knowledge in terms of obstacles.*

*It is not a question of considering external obstacles, such as the complexity or transience of phenomena, nor of incriminating the weakness of the senses or of the human spirit: it is in the very act of knowing, intimately, that hindrances and confusions appear, by a kind of functional necessity. It is there that we will show causes of stagnation and even regression, it is there that we will discern causes of inertia that we will call epistemological obstacles.*

In the didactics of mathematics, the concept of epistemological obstacle is resignified by Brousseau, in Barrantes (2006) where he conceptualizes the epistemological obstacle approaching it to the causes that lead to errors, as he points out: "The error is not only the effect of ignorance, uncertainty, but it is the effect of a previous knowledge, which, in spite of its interest or success, now reveals itself to be false or simply inadequate". Thus, when mentioning the epistemological obstacle, this author does not necessarily refer to erroneous knowledge; but to types of knowledge that are hindering the acquisition (construction) of a new one.

In both cases there is the difficulty of apprehending knowledge by a rational process, and thus applying these characteristics to Euclidean space and its conceptual generalization, in vector space, recognizing that it is necessary to make a historical review of the genesis of the concept of mathematical space. Therefore, the generated space can be considered as an epistemic obstacle that hinders and develops confusions for knowing, and the n-dimensional space additionally accuses the full category of didactic challenge for the acquisition or construction of mathematical knowledge.

Thus, the pedagogical process of constructing space falls fundamentally on constructivism as a common denominator, both in Piaget's proposal and in that of Pierre van Hiele and Dina van Diele-Geldof, as well as that of Brousseau, who can be situated in cognitive or socio-cultural constructivism, conceived as a dialectical process of the construction of reality.

## Results

In accordance with the above and with the history of mathematics, the appearance of the concept of vector and its generalization in the form of tensor, in the 19th century, generated alternatives of conceptual development for geometry and consequently the representation of space, with the appearance of the notion of vector space, whose development can be synthesized as follows for the construction of the didactic process:

First was the mathematical representation of the plane initiated by Descartes, which arises by associating it with the intersection of two orthogonal straight lines, which facilitates the Algebra-Geometry relationship for a full representation of mathematical objects and their numerical relationship of the straight line with the points and the real numbers, to thus define a set of two numbers that identify a position in the plane.

Then, it is recognized that the vector space is constructed from a fundamental idea of the intuition of the numerical calculation of addition and multiplication used in analytical geometry, as well as the direct relationship that exists when associating a number with a point, and that successively generates a numerical line, according to the Euclidean concept that a succession of points forms a line.

Returning to the previous idea of Descartes (2015) and in contrast, we can identify the notion that the whole is equal to the sum of its parts and also considering that multiplication is an abbreviated sum, then we have the elements to mathematically construct the n-dimensional space using addition and multiplication. As an initial part of the preliminary results of this research and as a result of the detailed analysis of the answers given to the interviews mentioned above, Table 1 was obtained:

| Onto-Epistemic Sieve | | | | |
|---|---|---|---|---|
| | Measure | Metrics | Space | Form |
| Epistemology | What is possible to conceive is perceived | Understanding the meaning | Euclidean Representation | Lack of intrinsic meaning |
| Ontology | Connection with conceptual interpretation | Idea of generalization | Qualitative assessment | Not stable conceptual invariants |

**Table 1** Onto-epistemic sieve
*Source: Own Elaboration*

In the table of our own elaboration, with the data obtained in the preliminary analysis, as previously pointed out was applied to the students who agreed to participate, in their answers, it is possible to perceive, that the concept of space is defined by the epistemology, identified with the subject, based on the intrinsic meaning, and by what he understands of mathematical knowledge, as well as the ontological, identified by the object that is determined in terms of a general concept with reality and its historical context, in this case without establishing conceptual invariants.

This makes it necessary to characterize the previous categories, mean, metric, space and form, in a finer form, to propose the genesis of a didactic strategy for the concept space, based on the historical development, in which the epistemic obstacle for the construction of vector space is identified, which is possible to consider as part of the fundamental teaching of vector space, as the process of decomposing a defined whole into its parts or its components, in this particular case of vector space by means of abbreviated multiplication.

## Conclusions

Clearly explain the results obtained and the possibilities for improvement. Each epistemic position tends to be displaced by another that better explains reality and allows the development of better concepts of geometric space of Euclidean characteristics. This is currently characterized in science by time defined in a functional relation of relative time-space, explained and formulated in a general way by tensor quantities.

Thus each epistemological posture represents an instant that is transformed into a historical moment of the integration of a general system of knowledge and its multiple vision, corresponding to the varied requirement of the problems of representation, explanation and formalization, having as a fundamental element the evolutionary continuity of scientific knowledge. The relationship of knowledge and its conceptual evolution lies in a continuous record of unfinished overcoming and transformations that denote the phases of intellectual development, implying the continuous development of thinking, systematized by the formalization and relativization of previous ideas.

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This fact gives enough guarantee to maintain a continuous development of mathematical thinking without meaning that it is necessary to reach an absolute perfection of knowledge, recognizing that this characteristic is not achieved at any point of human knowledge, which is why it is recognized that there are no didactic processes that are always successful, so it is not possible to establish so far an infallible formula that leads the learning process along a path traced by a path of achievements.

Positivism as a philosophical position promotes the establishment of general laws, allowing scientific knowledge to be classified and organized according to mathematical logic, as an integrated whole, leaving behind speculative processes of knowledge. Positivism considers that the development of humanity passes from the theological and metaphysical state to a positive state of knowledge.

Thus in positivism the concept of space, experienced a generalization based on a logical formalization as a certifying element of its subjective belonging, appearing different algebras that generalize this concept and that its didactic process is based on educational behaviorism as a didactic strategy.

On the contrary, instead of using logic, the concept of space can be developed as the construction of didactic strategies based on the concept of epistemological obstacle, developed in terms of a historical-conceptual reconstruction, in which the interrelation between geometric and algebraic reality must appear as a fundamental element, through the operations of addition and multiplication, as analyzed above.

In synthesis, this posture will allow the development of didactic strategies that adapt to the concept of space in a meaningful way for the student, starting from the process of construction of knowledge determined by fundamental ideas such as addition and product as basic operations that will determine the epistemological development and its possible obstacles of the idea of space.

In this way, a vein of conceptual inquiry would be generated, to fully answer the questions initially raised, and to answer the questions related to the concept, as was seen in the onto-epistemic sieve, and it is also possible to be complemented with the tools of didactic analysis to investigate the construction of mathematical space, such as those of Piaget, Pierre van Hiele and Dina van Diele-Geldof, Brousseau, Guershon Harel, Jean Luc Dorier and Ed Dubinsky's Action-Process-Object-Scheme (known as APOE), thus fulfilling the objectives proposed in this research proposal.

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