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Article

Analysis of solution methods for high order ordinary differential equations used in electrical circuits

Análisis de métodos de solución de ecuaciones diferenciales ordinarias de alto orden con aplicación en circuitos eléctricos

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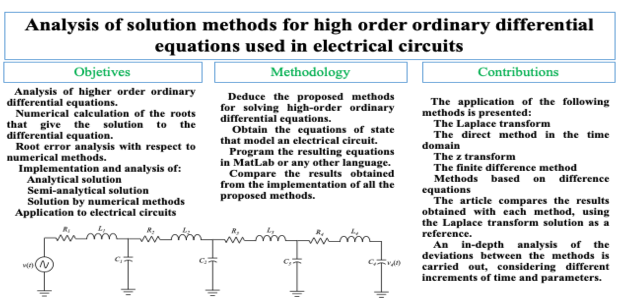
Abstract

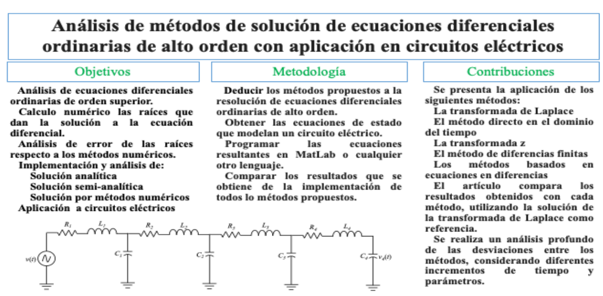
This article presents a detailed methodology for applying the most common methods in solving high-order ordinary differential equations in the simulation of electrical circuits. The methods analyzed include the Laplace transform, the direct method in the time domain, the z-transform, the finite difference method, and methods based on difference equations. A detailed development of each of these methods is provided, along with practical examples that demonstrate their application in specific electrical circuits. The examples include: a circuit modeled with a second-order ordinary differential equation, a circuit modeled with a third-order ordinary differential equation, and an electrical network whose modeling results in an eighth-order ordinary differential equation. The article compares the results obtained with each method, using the Laplace transform solution as a reference. A deep analysis of the deviations between the methods is conducted, considering different time steps and parameters, allowing conclusions to be drawn about the effectiveness and accuracy of each approach.



Differential equations, Laplace transform, Z-transform, Newton differences

Resumen

Este artículo presenta una metodología detallada para la aplicación de los métodos más comunes en la resolución de ecuaciones diferenciales ordinarias de alto orden en la simulación de circuitos eléctricos. Los métodos analizados incluyen la transformada de Laplace, el método directo en el dominio del tiempo, la transformada z, el método de diferencias finitas y los métodos basados en ecuaciones en diferencias. Se provee un desarrollo detallado de cada uno de estos métodos, acompañado de ejemplos prácticos que demuestran su aplicación en circuitos eléctricos específicos. Los ejemplos incluyen: un circuito modelado con una ecuación diferencial ordinaria de segundo orden, un circuito modelado con una ecuación diferencial ordinaria de tercer orden, una red eléctrica cuyo modelado resulta en una ecuación diferencial ordinaria de octavo orden. El artículo compara los resultados obtenidos con cada método, utilizando la solución de la transformada de Laplace como referencia. Se realiza un análisis profundo de las desviaciones entre los métodos, considerando diferentes incrementos de tiempo y parámetros, lo que permite llegar a conclusiones sobre la eficacia y precisión de cada enfoque.



Ecuaciones diferenciales, Transformada de Laplace, transformada z, Diferencias de Newton

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Introduction

Most of the physical phenomena that can be modeled with physical-mathematical formulations evolve in space and/or in time. Although there is a wide range of phenomena that only evolve in space or in time. In a natural way, some systems have a dynamic that stores or transfers energy in some of all its forms; mechanical, kinetic, potential, gravitational, acoustic, electrical, thermal, chemical, magnetic, nuclear, radiant, wind, solar, hydraulic or light. The physical-mathematical relationships of these systems are commonly modeled using differential equations; either partial if it depends on more than one independent spatial or space-temporal variable (Xiang et. al.) and/or ordinary if it only depends on a single independent variable, whether spatial or temporal. On the other hand, for the case of systems that depend on a single variable; the number of elements that store or transfer energy defines the order of the equation or the size of the systems of equations that model the physical phenomenon (Salas et. al.).

This work focuses on the modeling of concentrated electrical systems (Lathi, B. P.), that is, that only evolve in time; thus, the resulting model will always be a high order ordinary differential equation (HOODE).

The solution methods that have been developed were initially applied to simple or low-order equations; thus, intuitively the first of them was developed by Leonard Euler. It is worth mentioning that Isaac Newton gave mathematical formality to integro-differential calculus, which is why the first formal analytical formulation occurred with the formation of group theory for differential equations. Obviously, Pierre-Simón Laplace, a century later, established the entire theory for the solution of this type of equations using what is now known as the Laplace transform.

Years later, the z-transform theory was developed, which is based on the series by Pierre Alphonse Laurent. From this theory a method called difference equations was developed that takes advantage of the discrete plane scheme obtained from the application of the z-transform.

Solution methods for high-order ordinary differential equations

Normally an ordinary differential equation is solved analytically; the main reason is that this result is associated with a more precise solution (Saadeh, R. et. al.) and (Golmankhaneh, A. K., & Bongiorno, D.). However, in the case of high-order ordinary differential equations (HOODE), the analytical solution involves calculating the roots of a polynomial of the same order as that of the differential equation, and this has to be done numerically, with no option. This calculation is quite sensitive. In fact, it is not predictable to know if the final result will have more error due to the calculation of the roots or to other types of implementations (Marciniak et. al.). For this reason, this section deals with the analytical solution, semi-analytical methods and numerical methods for the solution of a HOODE.

Analytical methods

There are two traditional ways of solving a high-order ordinary differential equation, such as the Laplace transform, (Hsu, H. P.) and (Wilcox, D. J.), and its direct solution in the time domain by applying some method such as the indeterminate coefficients (Burden, L. R. & Faires, J. D.) and (Hoffman, J. D.). In this section, both methods are briefly described as well as the semi-analytical method of the z transform (Noda, T. & Ramirez, A.) and some numerical methods such as that of equations in differences or finite differences (Williams, P. W.), (Kinkaid D. R. & Hayes L. J.), (Smith G. D.) and (Strikwerda J.).

Laplace transform

Consider the HOODE with constant coefficients of the form

$$\sum_{k=0}^{N} a_k w^{(k)} = f(t) \quad (1)$$

with $w^m(0) = w_0^m \quad 0 \le m \le M-1$.

The Laplace transform of (1) has the following structure:

$$\sum_{k=0}^{N}\left(a_k s^k\right)W(s) - \sum_{k-0}^{N}\left(\sum_{j}^{k-1} s^{k-j-1} a_k w^{(j)}(0)\right) = F(s) \quad (2)$$

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Taking into account the initial conditions and clearing $W(s)$ from (2) we arrive at,

$$W(s)=\frac{F(s)+\sum_{k-0}^{N}\left(\sum_{j}^{k-1}s^{k-j-1}a_k w^{(j)}(0)\right)}{\sum_{k=0}^{N}\left(a_k s^k\right)} \tag{3}$$

The solution is obtained by algebraically expanding (3) to have simple elements to which the inverse Laplace transform is applied and arrive at a solution of the form,

$$w(t)=L^{-1}\{W(s)\} \tag{4}$$

Time domain solution

The direct solution of (1) is divided into two parts: in the first part the equation is equal to zero and it is proposed that the solution be of the type (homogeneous solution),

$$w(t)=w_t=e^{rt} \tag{5}$$

Thus, we have that the successive derivatives are of the form,

$$w_t^{(k)}=r^k e^{rt} \tag{6}$$

For $0\le k\le N$, by substituting (1) the characteristic polynomial or auxiliary equation is obtained,

$$\sum_{k=0}^{N}a_k r^k e^{rt}=e^{rt}\sum_{k=0}^{N}a_k r^k=0 \tag{7}$$

Since $e^{rt}\neq 0$, then the sum is necessarily equal to zero; Therefore, by means of the fundamental theorem of algebra, there are n solutions (roots) to the HOODE grouped in the following way,

$$w_t=\sum_{k=0}^{N}c_k r^{r_k t}=0 \tag{8}$$

The second part of the solution has to do with the $f(t)$ of the equation (particular solution). Here a function of the same type is proposed as $f(t)$ and the coefficients that adjust it are found. The sum of both solutions, homogeneous and particular, make up the total solution of the differential equation.

Semi-Analytical methods

It can be said that the z-transform method is semi-analytical since it starts from an analytical methodology until it reaches a numerical representation. The transformation technique is an important tool in the analysis of signals and linear systems that are invariant in time (ITLS). The z transform provides a means of characterizing ITLS and their response to various signals by the positions of their poles and zeros. The z-transform of a discrete signal in time is defined as the power series as,

$$X(z)=\sum_{n=-\infty}^{\infty}x(n)z^{-n} \tag{9}$$

where z is a complex variable denoted by $z=Ae^{j\omega}$. The previous relationship is called direct z-transform, since it transforms the signal $x(n)$ in the complex plane $X(z)$,

$$X(z)=Z\{x(n)\} \tag{10}$$

Note that the z-transform is an infinite power series, therefore, it only exists for those values of z for which the series converges.

To carry out the inverse transformation of an equation in the z-plane, the partial fraction technique will be used, so we can express the function $X(z)$, as a linear combination,

$$X(z)=\sum_{k=1}^{N}a_k X_k(z), \tag{11}$$

where $X_k(z)$ are expressions whose inverse transformations are $x_k(n)$. If such decomposition is possible, then $x(n)$ it is the inverse z-transform of $X(z)$, by linear combination of,

$$x(n)=\sum_{k=1}^{N}a_k x_k(n) \tag{12}$$

This method is particularly useful if $X(z)$ it is a rational function, that is

$$X(z)=\frac{N(z)}{D(z)} \tag{13}$$

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Where it is necessary that the denominator be of the form $D(z) = 1 + a_1 z^{-1} + L + a_N z^{-N}$.

For simplification it is multiplied $D(z)$ by the one z^N where it $-N$ corresponds to the greatest negative power, this in order not to have negative powers in the denominator, so we have,

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{N-k-1}}{z^N + \sum_{k=1}^{N} a_k z^{N-k}} \quad (14)$$

For this purpose, we first decompose $D(z)$ into factors containing the poles p_k of $X(z)$.

We have the cases of distinct poles, repeated poles, and conjugated complex poles; in this way, for the case where the poles are different, we have

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} \quad (15)$$

and the inverse z-transform of $X_k(z)$ is obtained by equivalence $Z^{-1}\{X_k(z)\} = (p_k)^n u(n)$, therefore

$$x(n) = u(n) \sum_{k=1}^{N} A_k (p_k)^n \quad (16)$$

For the case where we have multiplicity poles, the inverse transform is of the form,

$$Z^{-1}\left\{\frac{pz^{-1}}{(1 - pz^{-1})^k}\right\} = n^{k-1} p^n u(n) \quad (17)$$

In the case where we have some complex conjugated poles then complex exponentials are produced; However, if the signal $x(n)$ is real, it is possible to reduce said terms in real components; if we suppose that for some j between 1 and N they are had p_j and p_j^* in such a way that,

$$x_j(n) = \left[A_j (p_j)^n + A_j^* (p_j^*)^n\right] u(n) \quad (18)$$

and its combination in real components is,

$$x_j(n) = 2|A_j|(r_j)^n \cos(nb_j + a_j) u(n) \quad (19)$$

where $A_j = |A_j| e^{ia_j}$ and $p_j = r_j e^{ib_j}$. Thus, each z-domain conjugated complex pair produces a causal sinusoidal signal with an exponential envelope.

For the case where we have a differential equation of the type,

$$\sum_{k=1}^{N} a_k \frac{d^k w}{dt^k} = f(t) \quad (20)$$

the Laplace transform is applied first and the transition $W(s) \to W(z)$ and $s \to z$ is made by means of the trapezoidal or tustin rule $Z(s) = \frac{2}{\Delta t} \frac{z-1}{z-1}$ so, we have,

$$\mathrm{W}(\mathrm{z}) = \frac{\sum_{m=0}^{M} \mathrm{B}_m \mathrm{z}^{-m}}{\sum_{k=0}^{N} \mathrm{A}_k \mathrm{z}^{-k}} = \frac{\beta_0 + \mathrm{z}^{-1}\beta_1 + \cdots + \mathrm{z}^{-M}\beta_M}{\alpha_0 + \mathrm{z}^{-1}\alpha_1 + \cdots + \mathrm{z}^{-N}\alpha_N} \quad (21)$$

and thus, the inverse z-transform is found as $w(n) = Z^{-1}\{W(z)\}$

Numerical methods

The numerical implementation of a high-order ordinary differential equation is done through finite differences; the first derivative is approximated with,

$$w_n' = \frac{w_{n+1} - w_n}{h} \quad (22)$$

The above for a given value of *h*. From (23) a recursive form for a derivative of order *N* can be obtained as follows,

$$w^{(N)} = \sum_{k=0}^{N} \binom{N}{k} \frac{(-1)^k}{h^N} w_{n+N-k} \quad (23)$$

A linear differential equation of order N has the form,

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$$\sum_{k=0}^{N} a_k w^{(k)}(t) = f(t) \tag{24}$$

so, the difference scheme for this equation is,

$$\sum_{k=0}^{N} \sum_{j=0}^{k} \binom{k}{j} \frac{a_k (-1)^j}{h^k} w_{n+k-j} = f_n \tag{25}$$

clearing for w_{n+N} then the following relation is obtained

$$w_{n+N} = \frac{h^N}{a_N}\left(f_n - \sum_{k=0}^{N-1} \sum_{j=0}^{k} \binom{k}{j} \frac{a_k (-1)^j}{h^k} w_{n+k-j} - \sum_{j=1}^{N} \binom{N}{j} \frac{a_N (-1)^j}{h^N} w_{n+N-j} \right) \tag{26}$$

For a concurrence with an equation of order, initial conditions are necessary, which generates a unique solution to the differential equation (1).

Equations in differences

When constructing the mathematical model of some phenomenon, numerical and computational questions are interested, choosing a variable with discrete values. These data would be elements of a finite set, or failing that, a countable infinite. For this type of discrete deterministic models, the most appropriate mathematical tools to analyze them are the difference equations whose expression is of the type,

$$F\left(w_{n+N}, w_{n+N-1}, \ldots, w_{n+1}, w_n, n\right) = 0 \tag{27}$$

where the order of this equation is the value of the difference between the largest with the smallest of the indices of *F*. A difference equation is said to be linear with constant coefficients if it can be written as,

$$\sum_{k=0}^{N} a_k w(n+k) = f(n) \tag{28}$$

where $a_k \hat{I} R$. The objective is to find a function $w(n)$ that verifies that the equation for different values takes given values c_j, such that the solution is unique.

This is a fundamental system of solutions of the equation in differences, where the total solution is a linear combination of these, that is, if $\{w_1, w_2, \ldots, w_k\}$ are solutions, then the combination is solution too. So, we have,

$$w(n) = \sum_{j=1}^{k} c_j w_j(n) \tag{29}$$

For the equation in difference with $f(n) = 0$, a solution of the type $w(n) = r^n$ is sought, so when substituting the solution in the equation we have,

$$\sum_{k=o}^{N} a_k r^{n+k} = r^N \sum_{k=o}^{N} a_k r^k = 0 \tag{30}$$

which implies that,

$$\sum_{k=o}^{N} a_k r^k = 0 \tag{31}$$

where r^k are the roots of the equation in difference. These roots can be simple, repeated or complex conjugated. So, the solution with $f(n) = 0$ is,

$$w_h(n) = r_1^n p_{m-1}(n) + \sum_{k=m+1}^{N-2} c_k r_k^n + r^n\left(b_1 cos(qn) + b_2 sen(qn)\right) \tag{32}$$

with a single pair of complex conjugated roots $r_{1,2} = a \pm ib$ where $p = |\mathrm{r}|$ and $\theta = \arctan(b/a)$. There is also $P_{m-1}(n) = c_1 + c_2 n + L + c_m n^{m-1}$ where the degree depends on the multiplicity of the roots. To find the complete solution it is necessary to estimate the particular solution which depends on the nature of the function $f(n)$ defined in the equation in difference.

Equation in differences from z

The equation in differences starting from the transfer function in *z* is,

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$$H(z)=\frac{W(z)}{F(z)} = \frac{k_N z^N + k_{N-1}z^{N-1} + L + k_1 z + k_0}{c_M z^M + c_{M-1}z^{M-1} + L + c_1 z + c_0} \quad (33)$$

where M and N are the degree of the numerator and denominator respectively, generally if the bilinear transformation given by the trapezoidal rule is used, these are the same. So, multiplying by z^{-N} you get to,

$$W(z)\left[c_N + c_{N-1}z^{-1} + L + c_1 z^{-N+1} + c_0 z^{-N}\right] = F(z)\left[k_N + k_{N-1}z^{-1} + L + k_1 z^{-N+1} + k_0 z^{-N}\right] \quad (34)$$

therefore, in differences we have,

$$c_N W_n + c_{N-1}W_{n-1} + L + c_1 W_{n-N+1} + c_0 W_{n-N} = k_N F_n + k_{N-1}F_{n-1} + L + k_1 F_{n-N+1} + k_0 F_{n-N} \quad (35)$$

Equation in differences from finite newton differences

A linear differential equation $\sum_{k=0}^{N} a_k w^{(k)}(t) = f(t)$ of order N expressed in finite differences has the form,

$$\sum_{k=0}^{N}\sum_{j=0}^{k}\binom{k}{j}\frac{a_k(-1)^j}{h^k} w_{n+k-j} = f_n \quad (36a)$$

so, if we expand all the terms and group them algebraically, we will have an equation of the type

$$a_N w_n + a_{N-1}w_{n-1} + L + a_1 w_{n-N+1} + a_0 w_{n-N} = f_n \quad (36b)$$

Application examples

The solution of a high-order ordinary differential equation depends mainly on the solution of its characteristic polynomial. Because there are only methods for calculating roots in an analytical way up to 5rd order, then a strictly analytical exact solution can only be obtained up to that order.

In this section some cases will be analyzed. The first one will be a 2nd order equation with exact roots, in this way the analytical solution will be exact, strictly speaking. The second will be a 3rd order equation with arbitrary solutions. The third example will be a fictitious circuit that is represented by an 8th order equation.

In the first case the initial conditions are defined to implement all the methods, but for the other cases the initial conditions are obtained for the conditions of the case.

First example

This example is constructed from the roots, that means, the exact solution of the representative polynomial is known a priori, for that reason it is possible to construct the analytical solution in exact form. The HOODE is,

$$a_2\frac{d^2v}{dt^2} + a_1\frac{dv}{dt} + a_0 v = A\cos(\omega t) + B\sin(\omega t) + C$$

with

$a_2 = 1, a_1 = 6, a_0 = 8, A = 2, B = 3, C = 1, \omega = 2\pi f$, $f = 1/\pi$ and $T_{obs} = 8$ seconds. The initial conditions are $v(0) = 0$ and $v'(0) = 0$. The roots for this equation are $r_1 = -4$ and $r_2 = -2$.

Using the Laplace transforms

Applying the Laplace to the HOODE and taking into account the initial conditions, one obtain:

$$a_2\left(s^2V(s) - sv(0) - v'(0)\right) + a_1\left(sV(s) - v(0)\right) + a_0 V(s) = \frac{As}{s^2+\omega^2} + \frac{B\omega}{s^2+\omega^2} + \frac{C}{s}$$

So, the transfer function is,

$$H(s) = \frac{(A+C)s^2 + B\omega s + C\omega^2}{s^5 + 6s^4 + 12s^3 + 24s^2 + 32s}$$

Decomposing in partial fractions and solved we obtain

$$H(s) = \frac{0.175}{s+4} - \frac{0.125}{s+2} + \frac{0.125}{s} + \frac{-0.087-0.112i}{s-2i} + \frac{-0.087+0.112i}{s+2i}$$

Finally in time domain we have

$$v(t) = 0.175e^{-4t} - 0.125e^{-2t} + 0.125 + 2e^{-0t}\left(-0.087\cos(\omega t) + 0.112\sin(\omega t)\right)$$

Time domain solution

The HOODE is,

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$$a_2\frac{d^2v}{dt^2}+a_1\frac{dv}{dt}+a_0v=A\cos(\omega t)+B\sin(\omega t)+C$$

The proposed solution in time domain is as follows,
$v_T(t)=v_h(t)+v_p(t)=c_1e^{-4t}+c_2e^{-2t}+b_1cos(\omega t)+b_2sen(\omega t)+b_3$

Time domain particular solution

The particular proposed solution is as follows,

$$v_p(t)=b_1\cos(\omega t)+b_2\sin(\omega t)+b_3$$
$$v'_p(t)=-\omega b_1\sin(\omega t)+\omega b_2\cos(\omega t)$$
$$v''_p(t)=-\omega^2b_1\cos(\omega t)-\omega^2b_2\sin(\omega t)$$

Substituting in the ordinary differential equation (ODE), we obtain

$$\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}-\omega^2a_2+a_0 & a_1\omega & 0\\-\omega^2a_2+a_0 & -\omega^2a_2+a_0 & 0\\0 & 0 & a_0\end{bmatrix}^{-1}\begin{bmatrix}A\\B\\C\end{bmatrix}$$

which solution yield to

$$v_p(t)=-0.175\cos(\omega t)+0.225\sin(\omega t)+0.125$$

Time domain homogenous solution

Using this solution, we have
$$v_T(t)=c_1e^{-4t}+c_2e^{-2t}-0.175\cos(\omega t)+0.225\sin(\omega t)+0.125$$
$$v'_T(t)=-4c_1e^{-4t}-2c_2e^{-2t}+\omega 0.175\cos(\omega t)+\omega 0.225\sin(\omega t)$$

So, we construct the system of algebraic equations using the initial conditions as

$$\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1 & 1\\-4 & -2\end{bmatrix}^{-1}\begin{bmatrix}0.175\cos(0)-0.225\sin(0)-0.125\\-\omega 0.175\cos(0)-\omega 0.225\sin(0)\end{bmatrix}$$

Finally, we obtain $c_1=0.175$ and $c_2=-0.125$. So, the solution is:
$v_T(t)=0.175e^{-4t}-0.125e^{-2t}-0.175\cos(\omega t)+0.225\sin(\omega t)+0.125$

Z-transform solution

The solution in z-plane is constructed from $H(s)$, so we have $H(s)=\dfrac{(A+C)s^2+B\omega s+C\omega^2}{s^5+6s^4+12s^3+24s^2+32s}$

Then, by using MatLab to construct Hz; first it is used the instruction “Hs=tf([Ns], [Ds])” which creates a continuous-time transfer function SYS as Hs=tf([(A+C) ω*B ω 2C], [1 6 12 24 32 0]).

Having Hs, we use c2d to compute a discrete time model Hz, with sample time h, and with the trapezoidal rule (tustin) that approximates the continuous time model.

Hz = c2d(Hs,h,’tustin’)

$$H(z)=\frac{W(z)}{F(z)}=\frac{k_5z^5+k_4z^4+k_3z^3+k_2z^2+k_1z+k_0}{c_5z^5+c_4z^4+c_3z^3+c_2z^2+c_1z+c_0}$$

Finally, we use the function “residuez” to find the z-transform partial-fraction expansion of N(z)/D(z), so we obtain the residues, poles and direct term as

[r,p,Kp]=residuez(Hz.Numerator{1}, Hz.Denominator{1})

We construct the solution of the HOODE in z-domain as,
$$v(t)=\frac{r_1(p_1)^{ke}+r_2(p_2)^{ke}+r_3(p_3)^{ke}+r_4(p_4)^{ke}+r_5(p_5)^{ke}}{h}$$

with

$1\le ke\le n-1$ and $n=$ number of samples. The first sample is corrected with the direct term as, $v(1)=v(1)+\dfrac{Kp}{h}$.

Equation in differences solution from z

The solution begins with the transfer function of the left side of the differential equation, so we have

$$H(s)=\frac{1}{s^2+6s+8}$$

by using this, in z-domain we obtain

$$H(z)=\frac{W(z)}{F(z)}=\frac{\left[N_1z^2+N_2z+N_3\right]}{\left[D_1z^2+D_2z+D_3\right]}=\frac{z^{-2}}{z^{-2}}$$

Re-arranging we arrives to

$$V(z)\left[D_1+D_2z^{-1}+D_3z^{-2}\right]=F(z)\left[N_1+N_2z^{-1}+N_3z^{-2}\right]$$

So we have

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$$D_1V_n + D_2V_{(n-1)} + D_3V_{(n-2)} = N_1F_n + N_2F_{(n-1)} + N_3F_{(n-2)}$$

Particular solution

Solving for the particular proposed solution, we obtain

$$V_n = k_1\cos(M\Delta tn) + k_2\sin(M\Delta tn) + k_3$$
$$V_{(n-1)} = k_1\cos(M\Delta t(n-1)) + k_2\sin(M\Delta t(n-1)) + k_3$$
$$V_{(n-1)} = k_1\cos(M\Delta tn)\cos(M\Delta t) + k_1\sin(M\Delta tn)\sin(M\Delta t) + k_2\sin(M\Delta tn)\cos(M\Delta t) - k_2\cos(M\Delta tn)\sin(M\Delta t) + k_3$$
$$V_{(n-2)} = k_1\cos(M\Delta t(n-2)) + k_2\sin(M\Delta t(n-2)) + k_3$$
$$V_{(n-2)} = k_1\cos(M\Delta tn)\cos(2M\Delta t) + k_1\sin(M\Delta tn)\sin(2M\Delta t) + k_2\sin(M\Delta tn)\cos(2M\Delta t) - k_2\cos(M\Delta tn)\sin(2M\Delta t) + k_3$$

and

$$F_n = A\cos(M\Delta tn) + B\sin(M\Delta tn) + C$$
$$F_{(n-1)} = A\cos(M\Delta t(n-1)) + B\sin(M\Delta t(n-1)) + C$$
$$F_{(n-1)} = A\cos(M\Delta tn)\cos(M\Delta t) + A\sin(M\Delta tn)\sin(M\Delta t) + B\sin(M\Delta tn)\cos(M\Delta t) - B\cos(M\Delta tn)\sin(M\Delta t) + C$$
$$F_{(n-2)} = A\cos(M\Delta t(n-2)) + B\sin(M\Delta t(n-2)) + C$$
$$F_{(n-2)} = A\cos(M\Delta tn)\cos(2M\Delta t) + A\sin(M\Delta tn)\sin(2M\Delta t) + B\sin(M\Delta tn)\cos(2M\Delta t) - B\cos(M\Delta tn)\sin(2M\Delta t) + C$$

Substituting these functions in the equation we construct an algebraic system to obtain the coefficients as

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Where

$$a_{1,1} = D_1 + D_2\cos(M\Delta t) + D_3\cos(2M\Delta t)$$
$$a_{1,2} = -D_2\sin(M\Delta t) - D_3\sin(2M\Delta t)$$
$$a_{2,1} = D_2\sin(M\Delta t) + D_3\sin(2M\Delta t)$$
$$a_{2,2} = D_1 + D_2\cos(M\Delta t) + D_3\cos(2M\Delta t)$$
$$a_{1,3} = a_{2,3} = a_{3,1} = a_{3,2} = 0$$
$$a_{3,3} = D_1 + D_2 + D_3$$

And

$$b_1 = N_1A + N_2A\cos(M\Delta t) + N_3A\cos(2M\Delta t) - N_2B\sin(M\Delta t) - N_3B\sin(2M\Delta t)$$
$$b_2 = N_1B + N_2B\cos(M\Delta t) + N_3B\cos(2M\Delta t) + N_2A\sin(M\Delta t) + N_3A\sin(2M\Delta t)$$
$$b_3 = N_1C + N_2C + N_3C$$

solving the system, we arrive to

$$V_p(\mathrm{n}) = k_1\cos(M\Delta tn) + k_2\sin(M\Delta tn) + k_3$$

Homogeneous solution

The homogeneous solution is as follows

$$D_1V_n + D_2V_{(n-1)} + D_3V_{(n-2)} = 0$$

taking $\mathrm{V_n = t^n}$ and substituting in the previous equation we obtain $D_1t^n + D_2t^{n-1} + D_3t^{n-2} = 0$ so $t^{n-2}\left(D_1t^2 + D_2t^1 + D_3t^0\right) = 0$

the solution of this equation yield to

$$V_h(n) = c_1r_1^n + c_2r_2^n$$

Finally we have

$$V(\mathrm{n}) = c_1r_1^n + c_2r_2^n + k_1\cos(M\Delta tn) + k_2\sin(M\Delta tn) + k_3$$

By using two solutions to generate a system to determine the unknown coefficients we obtain

$$V(0) = \mathrm{c_1r_1^0 + c_2r_2^0} + k_1\cos(M\Delta t(0)) + k_2\sin(M\Delta t(0)) + k_3$$
$$V(1) = \mathrm{c_1r_1^1 + c_2r_2^1} + k_1\cos(M\Delta t(1)) + k_2\sin(M\Delta t(1)) + k_3$$

Finally, we construct an algebraic system as

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} r_1^0 & r_2^0 \\ r_1^1 & r_2^1 \end{bmatrix}^{-1} \begin{bmatrix} V(0) - k_1\cos(M\Delta t(0)) - k_2\sin(M\Delta t(0)) - k_3 \\ V(1) - k_1\cos(M\Delta t(1)) - k_2\sin(M\Delta t(1)) - k_3 \end{bmatrix}$$

The total solution is as follows

$$V(\mathrm{n}) = c_1r_1^n + c_2r_2^n + k_1\cos(M\Delta tn) + k_2\sin(M\Delta tn) + k_3$$

with $\mathrm{n = 1:N}$ and $\mathrm{N = number\ of\ samples}$

Equation in differences solution from finite NEWTON differences

From the HOODE,

$$a_2\frac{d^2v}{dt^2} + a_1\frac{dv}{dt} + a_0v = A\cos(\omega\mathrm{t}) + B\sin(\omega\mathrm{t}) + C$$

we begin with the Newton differences as

$$\frac{d^2v}{dt^2} = \frac{V_n - 2V_{n-1} + V_{n-2}}{h^2} \quad \text{and} \quad \frac{dv}{dt} = \frac{V_n - V_{n-1}}{h}$$

substituting into the equation we obtain

$$a_2\left(\frac{V_n - 2V_{n-1} + V_{n-2}}{h^2}\right) + a_1\left(\frac{V_n - V_{n-1}}{h}\right) + a_0V_n = A\cos(M\Delta tn) + B\sin(M\Delta tn) + C$$

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re-arranging we arrives to

$$\left(\frac{a_2}{h^2}+\frac{C_1}{h^1}+\frac{a_0}{h^0}\right)V_n+\left(-\frac{2a_2}{h^2}-\frac{a_1}{h^1}\right)V_{n-1}+\left(\frac{a_2}{h^2}\right)V_{n-2}=$$
$$A\cos\left(M\Delta tn\right)+B\sin\left(M\Delta tn\right)+C$$

or

$$Q_1V_n+Q_2V_{n-1}+Q_3V_{n-2}=A\cos\left(M\Delta tn\right)+B\sin\left(M\Delta tn\right)+C$$

Particular solution

Solving for the particular proposed solution, we obtain

$$V_n=k_1\cos\left(M\Delta tn\right)+k_2\sin\left(M\Delta tn\right)+k_3$$
$$V_{(n-1)}=k_1\cos\left(M\Delta t\left(n-1\right)\right)+k_2\sin\left(M\Delta t\left(n-1\right)\right)+k_3$$
$$V_{(n-1)}=k_1\cos\left(M\Delta tn\right)\cos\left(M\Delta t\right)+k_1\sin\left(M\Delta tn\right)\sin\left(M\Delta t\right)+$$
$$k_2\sin\left(M\Delta tn\right)\cos\left(M\Delta t\right)-k_2\cos\left(M\Delta tn\right)\sin\left(M\Delta t\right)+k_3$$
$$V_{(n-2)}=k_1\cos\left(M\Delta t\left(n-2\right)\right)+k_2\sin\left(M\Delta t\left(n-2\right)\right)+k_3$$
$$V_{(n-2)}=k_1\cos\left(M\Delta tn\right)\cos\left(2M\Delta t\right)+k_1\sin\left(M\Delta tn\right)\sin\left(2M\Delta t\right)+$$
$$k_2\sin\left(M\Delta tn\right)\cos\left(2M\Delta t\right)-k_2\cos\left(M\Delta tn\right)\sin\left(2M\Delta t\right)+k_3$$

Using these functions in the Newton difference equation we obtain a system like

$$\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}=\begin{bmatrix}a_{1,1}&a_{1,2}&a_{1,3}\\a_{2,1}&a_{2,2}&a_{2,3}\\a_{3,1}&a_{3,2}&a_{3,3}\end{bmatrix}^{-1}\begin{bmatrix}A\\B\\C\end{bmatrix}$$

where

$$a_{1,1}=Q_1+Q_2\cos\left(M\Delta t\right)+Q_3\cos\left(2M\Delta t\right)$$
$$a_{1,2}=-Q_2\sin\left(M\Delta t\right)-Q_3\sin\left(2M\Delta t\right)$$
$$a_{1,3}=0$$
$$a_{2,1}=Q_2\sin\left(M\Delta t\right)+Q_3\sin\left(2M\Delta t\right)$$
$$a_{2,2}=Q_1+Q_2\cos\left(M\Delta t\right)+Q_3\cos\left(2M\Delta t\right)$$
$$a_{2,3}=a_{3,1}=a_{3,2}=0$$

and $a_{3,3}=Q_1+Q_2+Q_3$

solving the system, we arrive to

$$V_p\left(\mathrm{n}\right)=k_1\cos\left(M\Delta tn\right)+k_2\sin\left(M\Delta tn\right)+k_3$$

Homogeneous solution

The homogeneous solution is as follows

$$Q_1V_n+Q_2V_{n-1}+Q_3V_{n-2}=0$$

taking $\mathrm{V_n}=\mathrm{t^n}$ and substituting in the previous equation we obtain $Q_1t^n+Q_2t^{n-1}+Q_3t^{n-2}=0$

so $t^{n-2}\left(Q_1t^2+Q_2t^1+Q_3t^0\right)=0$ the solution of this equation yield to $V_h\left(\mathrm{n}\right)=\mathrm{c_1r_1^n+c_2r_2^n}$

Finally we have

$$V\left(\mathrm{n}\right)=c_1r_1^n+c_2r_2^n+k_1\cos\left(M\Delta tn\right)+k_2\sin\left(M\Delta tn\right)+k_3$$

By using two solutions to generate a system to determine the unknown coefficients we obtain

$$V\left(0\right)=\mathrm{c_1r_1^0+c_2r_2^0}+k_1\cos\left(M\Delta t\left(0\right)\right)+k_2\sin\left(M\Delta t\left(0\right)\right)+k_3$$
$$V\left(1\right)=\mathrm{c_1r_1^1+c_2r_2^1}+k_1\cos\left(M\Delta t\left(1\right)\right)+k_2\sin\left(M\Delta t\left(1\right)\right)+k_3$$

Finally, we construct an algebraic system as

$$\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}r_1^0&r_2^0\\r_1^1&r_2^1\end{bmatrix}^{-1}\begin{bmatrix}V\left(0\right)-k_1\cos\left(M\Delta t\left(0\right)\right)-k_2\sin\left(M\Delta t\left(0\right)\right)-k_3\\V\left(1\right)-k_1\cos\left(M\Delta t\left(1\right)\right)-k_2\sin\left(M\Delta t\left(1\right)\right)-k_3\end{bmatrix}$$

The total solution is as follows

$$V\left(\mathrm{n}\right)=\mathrm{c_1r_1^n+c_2r_2^n}+k_1\cos\left(M\Delta tn\right)+k_2\sin\left(M\Delta tn\right)+k_3$$

with $1\le n\le N\quad and\quad N=number\ of\ samples$

Solving with different Δt

Table 1 shows the used Δt in this equation, this table numbered each time step, for example the 20th position is associated with Δt equal to 0.4 seconds.

Box 1

Table 1

Position of each used Δt.

Pos	Δt in sec	Pos	Δt in sec	Pos	Δt in sec
1	2e-07	8	4e-05	15	0.008
2	4e-07	9	8e-05	16	0.02
3	8e-07	10	0.0002	17	0.04
4	2e-06	11	0.0004	18	0.08
5	4e-06	12	0.0008	19	0.2
6	8e-06	13	0.002	20	0.4
7	2e-05	14	0.004	21	0.8

Own generation

Note: We denote with green the Δt for which we obtain the lowest error and with blue the second Δt in term of the obtained error.

Lowest error for each kind of simulation
Next error for each kind of simulation

From table 2 to 17 we show all the made texts to the proposed ordinary differential equation.

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Box 2

Table 2

MSE taking the Laplace solution like a reference - $MSE_{Error} = abs(sum((Laplace - Method).\hat{}2))$/Ns

Δt in sec	Time domain	Z-domain	Differences from Z	Differences from Newton
2e-07	1.5921e-30	NaN	1.377e-09	1.2894e-12
4e-07	1.5921e-30	NaN	8.4146e-11	4.7858e-15
8e-07	1.5921e-30	NaN	1.922e-12	1.6928e-14
2e-06	1.5922e-30	NaN	1.9879e-12	1.0192e-12
4e-06	1.5922e-30	NaN	1.0566e-14	8.9919e-13
8e-06	1.5921e-30	NaN	6.0996e-16	3.9757e-12
2e-05	1.5924e-30	Inf	6.8982e-17	2.3767e-11
4e-05	1.5921e-30	6.77e+185	4.5111e-19	9.5358e-11
8e-05	1.5923e-30	2.221e+81	3.6698e-19	3.8149e-10
0.0002	1.5918e-30	8.143e+22	7.8259e-18	2.384e-09
0.0004	1.5919e-30	608.15	1.2763e-16	9.5336e-09
0.0008	1.5923e-30	0.0023425	2.0453e-15	3.8117e-08
0.002	1.5907e-30	7.1112e-06	7.989e-14	2.3789e-07
0.004	1.5916e-30	8.0817e-10	1.278e-12	9.4933e-07
0.008	1.5936e-30	1.6423e-09	2.0442e-11	3.7794e-06
0.02	1.5865e-30	6.3859e-08	7.9781e-10	2.3288e-05
0.04	1.5846e-30	1.021e-06	1.2749e-08	9.0959e-05
0.08	1.5896e-30	1.6222e-05	2.0369e-07	0.00034666
0.2	1.5795e-30	0.0005917	7.9807e-06	0.0018626
0.4	1.5696e-30	0.0071934	0.00013146	0.0056763
0.8	1.6396e-30	0.040308	0.0023949	0.013309

Own generation

Box 3

Table 3

Area Error taking the Laplace solution like a reference - $AREA_{Error} = abs(trapz(Laplace) - trapz(Method))$

Δt in sec	Time domain	Z-domain	Differences from Z	Differences from Newton
2e-07	3.3307e-15	NaN	5.8501e-05	4.6979e-06
4e-07	3.5527e-15	NaN	6.4537e-05	1.1724e-07
8e-07	3.5527e-15	NaN	3.5474e-06	4.1911e-07
2e-06	3.1086e-15	NaN	5.2728e-06	1.4228e-06
4e-06	3.5527e-15	Inf	5.8488e-07	1.8001e-06
8e-06	3.5527e-15	NaN	1.4061e-07	3.7146e-06
2e-05	3.5527e-15	1.16e+163	4.5574e-08	9.1643e-06
4e-05	3.3307e-15	6.213e+92	2.8457e-09	1.8345e-05
8e-05	3.5527e-15	4.944e+40	2.2008e-09	3.6691e-05
0.0002	3.3307e-15	5.674e+11	5.8192e-10	9.1725e-05
0.0004	3.5527e-15	105.86	2.4506e-09	0.00018344
0.0008	3.3307e-15	0.28607	1.0252e-08	0.00036683
0.002	3.5527e-15	0.016371	6.3911e-08	0.00091676
0.004	3.5527e-15	0.0001480	2.5434e-07	0.0018324
0.008	3.5527e-15	2.515e-05	1.007e-06	0.0036602
0.02	3.1086e-15	0.0001467	6.1051e-06	0.0091137
0.04	3.3307e-15	0.0005948	2.3251e-05	0.018088
0.08	3.5527e-15	0.0024562	8.4909e-05	0.035502
0.2	3.3307e-15	0.01734	0.00043589	0.081493
0.4	3.1086e-15	0.080994	0.0016637	0.12587
0.8	3.1086e-15	0.20659	0.0079006	0.038226

Own generation

Box 4

Table 4

Percent Error taking the Laplace solution like a reference
$\%_{Error} = abs(sum((Lap - Method)/max(Lap)))$/Ns

Δt in sec	Time domain	Z-domain	Differences from Z	Differences from Newton
2e-07	1.0409e-15	NaN	1.7834e-05	1.4321e-06
4e-07	1.0409e-15	NaN	1.9674e-05	3.5739e-08
8e-07	1.041e-15	NaN	1.0814e-06	1.2776e-07
2e-06	1.041e-15	NaN	1.6074e-06	4.3374e-07
4e-06	1.041e-15	Inf	1.783e-07	5.4876e-07
8e-06	1.0409e-15	NaN	4.2865e-08	1.1324e-06
2e-05	1.041e-15	3.55e+162	1.3893e-08	2.7937e-06
4e-05	1.0409e-15	1.895e+92	8.6749e-10	5.5924e-06
8e-05	1.0405e-15	1.506e+40	6.7092e-10	1.1185e-05
0.0002	1.0413e-15	1.730e+11	1.7746e-10	2.7963e-05
0.0004	1.0403e-15	32.275	7.4757e-10	5.5924e-05
0.0008	1.0404e-15	0.087199	3.1292e-09	0.00011184
0.002	1.0383e-15	0.0049899	1.9546e-08	0.00027957
0.004	1.0392e-15	4.5089e-05	7.804e-08	0.000559
0.008	1.0394e-15	7.528e-06	3.1101e-07	0.0011174
0.02	1.0277e-15	4.2586e-05	1.9243e-06	0.0027881
0.04	1.0186e-15	0.0001643	7.5938e-06	0.0055527
0.08	1.0132e-15	0.0006169	2.9947e-05	0.010978
0.2	9.6246e-16	0.0034557	0.00019556	0.025719
0.4	8.7295e-16	0.015162	0.0010144	0.043009
0.8	7.1562e-16	0.059835	0.0057003	0.029467

Own generation

NOTE: The results that are highlighted are those with the lowest error for each presented method.

Box 5

Table 5

MSE using f=60 Hz
$MSE_{Error} = abs(sum((Laplace - Method).\hat{}2))$/Ns

Method	Time domain	Z-domain	Differences from Z	Differences Newton
Error	6.1124e-33	3.5386e-10	2.9712e-17	4.245e-14
Position	15	15	7	4
Error	6.1493e-33	4.0871e-10	1.1584e-16	5.5324e-14
Position	13	16	8	2

Own generation

Box 6

Table 6

Area Error using f=60 Hz

$$AREA_{Error} = abs(trapz(Laplace) - trapz(Method))$$

Method	Time domain	Z-domain	Differences from Z	Differences Newton
Error	4.4409e-16	8.5023e-08	2.6795e-08	7.7232e-07
Position	15	14	7	4
Error	4.4409e-16	1.1263e-07	3.9154e-08	1.0162e-06
Position	18	15	8	2

Own generation

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Box 7

Table 7

Percent Error using f=60 Hz
$\%_{Error} = abs(sum((Lap - Method)/max(Lap)))$/Ns

Method	Time domain	Z-domain	Differences from Z	Differences Newton
Error	5.8716e-16	6.3457e-08	2.679e-08	7.7216e-07
Position	15	14	7	4
Error	5.8937e-16	6.8251e-08	3.9146e-08	1.016e-06
Position	13	15	8	2

Own generation

Box 8

Table 8

MSE using A=2e20, B=3e20 and C=1e20 - $MSE_{Error} = abs(sum((Laplace - Method).^2))$/Ns

Method	Time domain	Z-domain	Differences from Z	Differences Newton
Error	1.6838e+10	8.0817e+30	3.6698e+21	4.7858e+25
Position	13	14	9	2
Error	1.6838e+10	1.6423e+31	4.5111e+21	1.6928e+26
Position	15	15	8	3

Box 9

Table 9

Area Error using A=2e20, B=3e20 and C=1e20 - $AREA_{Error} = abs(trapz(Laplace) - trapz(Method))$

Method	Time domain	Z-domain	Differences from Z	Differences Newton
Error	3.2768e+05	2.515e+15	5.8189e+10	1.1724e+13
Position	2	15	10	2
Error	3.2768e+05	1.4674e+16	2.2008e+11	4.1911e+13
Position	6	16	9	3

Box 10

Table 10

Percent Error using A=2e20, B=3e20 and C=1e20 - $\%_{Error} = abs(sum((Lap - Method)/max(Lap)))$/Ns

Method	Time domain	Z-domain	Differences from Z	Differences Newton
Error	7.9708e-16	7.528e-06	1.7745e-10	3.574e-08
Position	21	15	10	2
Error	9.3418e-16	4.2586e-05	6.7091e-10	1.2776e-07
Position	20	16	9	3

Box 11

Table 11

MSE with A=2e20, B=3e20, C=1e20 and F=60 Hz
$MSE_{Error} = abs(sum((Laplace - Method).^2))$/Ns

Method	Time domain	Z-domain	Differences from Z	Differences Newton
Error	5.8712e+07	3.5386e+30	2.9712e+23	4.245e+26
Position	16	15	7	4
Error	5.9091e+07	4.0871e+30	1.1584e+24	5.5325e+26
Position	17	16	8	2

Box 12

Table 12

Area Error using A=2e20, B=3e20, C=1e20 and F=60 Hz
$AREA_{Error} = abs(trapz(Laplace) - trapz(Method))$

Method	Time domain	Z-domain	Differences from Z	Differences Newton
Error	49152	8.5023e+12	2.6795e+12	7.7232e+13
Position	1	14	7	4
Error	49152	1.1263e+13	3.9154e+12	1.0162e+14
Position	4	15	8	2

Box

Table 13

Percent Error A=2e20, B=3e20, C=1e20 and F=60 Hz
$\%_{Error} = abs(sum((Lap - Method)/max(Lap)))$/Ns

Method	Time domain	Z-domain	Differences from Z	Differences Newton
Error	5.7701e-16	6.3457e-08	2.679e-08	7.7216e-07
Position	16	14	7	4
Error	5.7985e-16	6.8251e-08	3.9146e-08	1.016e-06
Position	17	15	8	2

Box 14

Table 14

Errors using different values for A, B, C and F
A=2, B=3, C=1 and F=1/pi Hz.

Kind of error	Time domain	Z-domain	Differences from Z	Differences Newton
RMS_{Error}	20	14	9	2
	19	15	8	3
$AREA_{Error}$	20	15	10	2
	4	16	9	3
$\%_{Error}$	21	15	10	2
	20	16	9	3

Box 15

Table 15

Errors using different values for A, B, C and F
A=2, B=3, C=1 and F=60 Hz.

Kind of error	Time domain	Z-domain	Differences from Z	Differences Newton
RMS_{Error}	15	15	7	4
	13	16	8	2
$AREA_{Error}$	15	14	7	4
	18	15	8	2
$\%_{Error}$	15	14	9	2
	13	15	8	3

Box 16

Table 16

Errors using different values for A, B, C and F
A=2e20, B=3e20, C=1e20 and F=1/pi Hz.

Kind of error	Time domain	Z-domain	Differences from Z	Differences from Newton
RMS_{Error}	13	14	9	2
	15	15	8	3
$AREA_{Error}$	2	15	10	2
	6	16	9	3
$\%_{Error}$	21	15	10	2
	20	16	9	3

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Box 17

Table 17

Errors using different values for A, B, C and F
A=2e20, B=3e20, C=1e20 and F=60 Hz.

Kind of error	Time domain	Z-domain	Differences from Z	Differences from Newton
RMS_{Error}	16	15	7	4
	17	16	8	2
$AREA_{Erro}$	1	14	7	4
	4	15	8	2
$\%_{Error}$	16	14	7	4
	17	15	8	2

Findings

Errors are calculated according to the equation shown in the header of each table. In the case of MSE, the difference between samples is calculated, squared and added; For the case of area error, the area under the curve is calculated with the trapezoidal rule and the absolute difference is obtained. Thus, analyzing the results obtained, the following was found:

1. The solution in time compared to Laplace are practically identical, the differences are due to the fact that numerical computation in binary is of finite length and therefore has an inherent error.

2. The foregoing is noticeable when seeing the RMS error in which it decreases because as the delta t increases, the samples decrease, but the area error is correspondingly similar for all cases.

3. The Z transform depends on the delta t used, so for the case of a very small delta t the methodology is indeterminate, as the delta t grows it stabilizes numerically until it reaches its maximum precision, then as the delta t increases it loses precision.

4. The difference equation, starting from the Z transform, increases precision as delta t grows until it reaches a point where more and more precision is lost.

If all the results are analyzed, it is concluded that, at least for this equation, there is no delta t that is suitable for all implementations. Figure 1 and 2 shows graphically one of these results.

Box 18

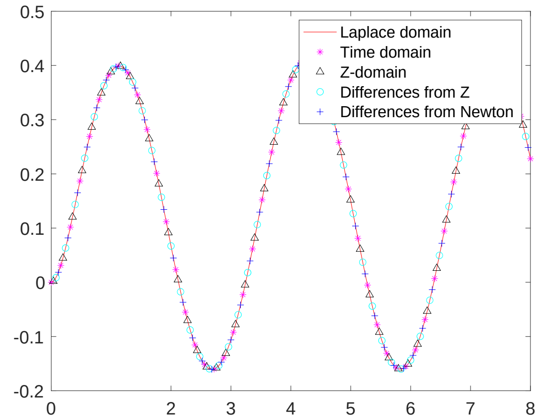

Figure 1

Solution by using A=2, B=3, C=1 and F=1/pi Hz

Box 19

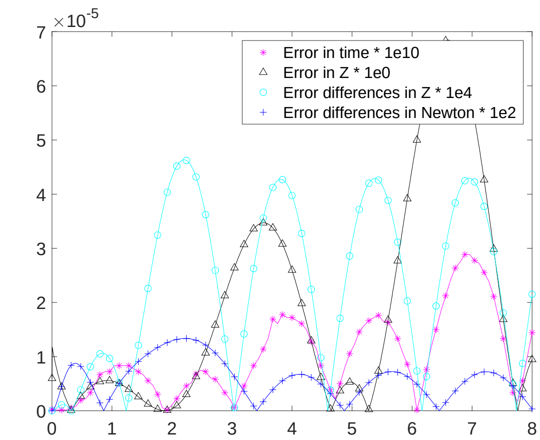

Figure 2

Error by using A=2, B=3, C=1 and F=1/pi Hz

Second example

This example (figure 3) is constructed from the roots, that means, the exact solution of the representative polynomial is known a priori, for that reason it is possible to construct the analytical solution in exact form. The HOODE is,

$$\frac{d^3v}{dt^3}+C\frac{d^2v}{dt^2}+D\frac{dv}{dt}+Ev=A\cos\left(\omega t\right)+B\sin\left(\omega t\right)+C$$

with $C=200$, $D=4040004$, $E=40000$, $A=-10000w^2$, $B=-2000000w$, $C=0$ and $T_{obs}=4$ seconds. The initial conditions, $v=0$, $v'=0$ and $v''=0$

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Box 20

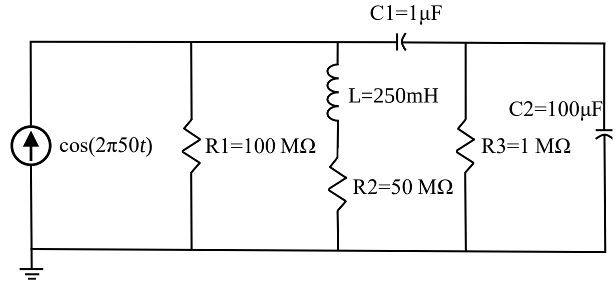

Figure 3
Electrical network used in this example

Using the Laplace transforms

Applying the Laplace to the HOODE and taking into account the initial conditions, one obtains:

$$s^3 + Cs^2 + Ds + E = \frac{As}{s^2+\omega^2} + \frac{B\omega}{s^2+\omega^2}$$

So, the transfer function is,

$$H(s) = \frac{As + B\omega}{s^5 + k_1 s^4 + k_2 s^3 + k_3 s^2 + k_4 s + k_5}$$

with $k_1 = 200$, $k_2 = 4182126$, $k_3 = 28464460$, $k_4 = 574174674126$, and $k_5 = 5684892135$.

Decomposing in partial fractions and solved we obtain

$$H(s) = \frac{-0.0182-0.0889i}{s+(-99.99+2007.48i)} + \frac{-0.0182+0.0889i}{s+(-99.99-2007.48i)} + \frac{-0.4950}{s+(-0.0099)} + \frac{0.2657+0.4784i}{s-376.99i} + \frac{0.2657-0.4784i}{s+376.99i}$$

Finally in time domain, with $r_1 = -99.9950 + 2007.4867i$, $r_2 = -99.9950 - 2007.4867i$ and $r_3 = -0.0099$, we have

$$v(t) = (-0.0182-0.0889i)e^{-r_1 t} + (-0.0182+0.0889i)e^{-r_2 t} + (-0.4950)e^{-r_3 t} + 2e^{-0t}(0.265791\cos(\omega t) - 0.478450\sin(\omega t))$$

Time domain solution

The HOODE is,

$$\frac{d^3v}{dt^3} + C\frac{d^2v}{dt^2} + D\frac{dv}{dt} + Ev = A\cos(\omega t) + B\sin(\omega t)$$

The proposed solution in time domain is as follows,

$$v_T(t) = c_1 e^{-r_1 t} + c_2 e^{-r_2 t} + c_3 e^{-r_3 t} + b_1\cos(\omega t) + b_2\sin(\omega t)$$

Time domain particular solution

The particular proposed solution is as follows,

$$v_p(t) = b_1\cos(\omega t) + b_2\sin(\omega t)$$

which solution yield to

$$v_p(t) = 0.53158\cos(\omega t) - 0.9569\sin(\omega t)$$

Time domain homogenous solution

Using this solution, we have the total solution as:

$$v_T(t) = c_1 e^{-r_1 t} + c_2 e^{-r_2 t} + c_3 e^{-r_3 t} + 0.5315\cos(\omega t) - 0.9569\,\mathrm{sen}(\omega t)$$

by using this proposed solution with $r_1 = -99.99 + 2007.5i$, $r_2 = -99.99 - 2007.5i$, $r_3 = -0.0099$, and the initial conditions, to generate the algebraic system, we obtain: $c_1 = -0.0182 - 0.0889$, $c_1 = -0.0182 + 0.0889$ and $c_3 = -0.4950$; so, the final solution is,

$$v_T(t) = (-0.0182-0.0889)e^{-r_1 t} + (-0.0182+0.0889)e^{-r_2 t} - 0.4950e^{-r_3 t} + 0.5315\cos(\omega t) - 0.9569\sin(\omega t)$$

Z-transform solution

The solution in z-plane is constructed from $H(s)$, so we have

$$H(s) = \frac{As + B\omega}{s^5 + k_1 s^4 + k_2 s^3 + k_3 s^2 + k_4 s + k_5}$$

Then, by using MatLab to construct Hz; first it is used the instruction “Hs=tf([Ns], [Ds])” which creates a continuous-time transfer function SYS as,

- Hs=tf([A w*B], [k_1 k_2 k_3 k_4 k_5])

Having Hs, we use c2d to compute a discrete time model Hz, with sample time $h = 8x10^{-5}$, and with the trapezoidal rule that approximates the continuous time model.

- Hz = c2d(Hs,h,’tustin’)

So, we have

$$H(z) = \frac{W(z)}{F(z)} = \frac{-3.61e^{-9}z^5 - 1.09e^{-8}z^4 - 7.45e^{-9}z^3 + 6.88e^{-9}z^2 + 1.06e^{-8}z + 3.55e^{-9}}{z^5 - 4.958z^4 + 9.858z^3 - 9.826z^2 + 4.911z - 0.9842}$$

Finally, we use the function “residuez” to find the z-transform partial-fraction expansion of N(z)/D(z), so we obtain the residues, poles and direct term as

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[r,p,Kp]=residuez(Hz.Numerator{1}, Hz.Denominator{1})

We construct the solution of the HOODE in z-domain as,

$$v(t)=\frac{r_1(p_1)^{ke}+r_2(p_2)^{ke}+r_3(p_3)^{ke}+r_4(p_4)^{ke}+r_5(p_5)^{ke}}{h}$$

with $r_1=2.125e^{-5}+3.826e^{-5}i$, $r_2=2.125e^{-5}-3.826e^{-5}i$, $r_3=-3.960e^{-5}$, $r_4=-1.457e^{-6}-7.068e^{-6}i$, $r_5=-1.457e^{-6}+7.068e^{-6}i$, $p_1=0.999+0.030$, $p_2=0.999-0.030$, $p_3=1$, $p_4=0.979+0.158$, $p_5=0.979-0.158$, $Kp=0$, $1\le ke\le n-1$, and $n=$ number of samples. The first sample is corrected with the direct term as, $v(1)=v(1)+\frac{Kp}{h}$.

Equation in differences solution from z

The solution begins with the transfer function of the left side of the differential equation, so we have

$$H(s)=\frac{1}{s^3+200s^2+4040004s+40000}$$

by using $H(s)$ with $h=4x10^{-6}$, in z-domain we obtain

$$H(z)=\frac{V(z)}{F(z)}$$

$$=\frac{\left[7.997e^{-18}z^3+2.399e^{-17}z^2+2.399e^{-17}z+7.997e^{-18}\right]}{\left[z^3-2.999z^2+2.998z-0.999\right]}\frac{z^{-3}}{z^{-3}}$$

that is

$$H(z)=\frac{V(z)}{F(z)}=\frac{\left[N_1+N_2z^{-1}+N_3z^{-2}+N_4z^{-3}\right]}{\left[D_1+D_2z^{-1}+D_3z^{-2}+D_4z^{-3}\right]}$$

Re-arranging we arrive to

$$V(z)\left[D_1+D_2z^{-1}+D_3z^{-2}+D_4z^{-3}\right]=F(z)\left[N_1+N_2z^{-1}+N_3z^{-2}+N_4z^{-3}\right]$$

So we have

$$D_1V_n+D_2V_{(n-1)}+D_3V_{(n-2)}+D_4V_{(n-3)}=N_1F_n+N_2F_{(n-1)}+N_3F_{(n-2)}+N_4F_{(n-3)}$$

Particular solution

Solving for the particular proposed solution, we obtain

$$V_n=k_1\cos(M\Delta tn)+k_2\sin(M\Delta tn)+k_3$$

$$V_{(n-1)}=k_1\cos(M\Delta t(n-1))+k_2\sin(M\Delta t(n-1))+k_3$$

$$V_{(n-2)}=k_1\cos(M\Delta t(n-2))+k_2\sin(M\Delta t(n-2))+k_3$$

and

$$F_n=A\cos(M\Delta tn)+B\sin(M\Delta tn)+C$$

$$F_{(n-1)}=A\cos(M\Delta t(n-1))+B\sin(M\Delta t(n-1))+C$$

$$F_{(n-2)}=A\cos(M\Delta t(n-2))+B\sin(M\Delta t(n-2))+C$$

Substituting these functions in the equation we construct an algebraic system to obtain the coefficients. So, we arrive to

$$V_p(n)=0.53158\cos(M\Delta tn)-0.9569\sin(M\Delta tn)$$

Homogeneous solution

The homogeneous solution is as follows

$$D_1V_n+D_2V_{(n-1)}+D_3V_{(n-2)}+D_4V_{(n-3)}=0$$

taking and substituting in the previous equation we obtain

$$D_1t^n+D_2t^{n-1}+D_3t^{n-2}+D_4t^{n-3}=0$$

so

$$t^{n-3}\left(D_1t^3+D_2t^2+D_3t^1+D_4t^0\right)=0$$

the solution of this equation yield to

$$V_h(n)=c_1r_1^n+c_2r_2^n+c_3r_3^n$$

with $r_1=-99.9950+2007.4867$, $r_2=-99.9950-2007.4867$ and $r_3=-0.0099$.

Finally, we have

$$V(n)=c_1r_1^n+c_2r_2^n+c_3r_3^n$$
$$+0.53158\cos(M\Delta tn)-0.9569\sin(M\Delta tn)$$

By using three solutions to generate a system to determine the unknown coefficients we obtain

$$V(n)=(-0.018279-0.088938i)r_1^n+(-0.018279+0.088938i)r_2^n$$
$$-0.49502r_3^n+0.53158\cos(M\Delta tn)-0.9569\sin(M\Delta tn)$$

with $n=1:N$ and $N=$ number of samples

Equation in differences solution from finite Newton differences

From the HOODE,

$$a_3\frac{d^3v}{dt^3}+a_2\frac{d^2v}{dt^2}+a_1\frac{dv}{dt}+a_0v=A\cos(\omega t)+B\sin(\omega t)$$

substituting into the equation Newton difference, with $h=8x10^{-7}$, we obtain

$$Q_1V_n+Q_2V_{n-1}+Q_3V_{n-2}+Q_4V_{n-3}=A\cos(M\Delta tn)+B\sin(M\Delta tn)$$

with

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$Q_1 = 1953442550005040128$
$Q_2 = -5860005050005001216$
$Q_3 = 5859687500000001024$ and
$Q_4 = -1953125000000000256$

Particular solution

Solving for the particular proposed solution, we obtain
$V_n = k_1 \cos(M\Delta tn) + k_2 \sin(M\Delta tn)$

Using these functions in the Newton difference equation we arrive to

$$V_p(\mathrm{n}) = \frac{2780}{5231}\cos(M\Delta tn) - \frac{645}{674}\sin(M\Delta tn)$$

Homogeneous solution

The homogeneous solution is as follows
$Q_1V_n + Q_2V_{n-1} + Q_3V_{n-2} + Q_4V_{n-3} = 0$
taking $V_n = t^n$ and substituting in the previous equation we obtain
$Q_1t^n + Q_2t^{n-1} + Q_3t^{n-2} + Q_4t^{n-3} = 0$

so

$$t^{n-3}\left(Q_1t^3 + Q_2t^2 + Q_3t^1 + Q_4t^0\right) = 0$$

the solution of this equation yield to

$$V_h(n) = c_1\left(\frac{12110}{12111} + \frac{74}{46085}i\right)^n + c_2\left(\frac{12110}{12111} - \frac{74}{46085}i\right)^n + c_3(1)^n$$

Finally we have

$$V(n) = c_1r_1^n + c_2r_2^n + c_3r_3^n + \frac{2780}{5231}\cos(M\Delta tn) - \frac{645}{674}\sin(M\Delta tn)$$

By using two solutions to generate a system to determine the unknown coefficients we obtain the solution as follows

$$V(n) = \left(-\frac{39}{2117} - \frac{290}{3261}i\right)\left(\frac{12110}{12111} + \frac{74}{46085}i\right)^n + \left(-\frac{39}{2117} + \frac{290}{3261}i\right)\left(\frac{12110}{12111} - \frac{74}{46085}i\right)^n - \frac{504}{1019}(1)^n + \frac{2780}{5231}\cos(M\Delta tn) - \frac{645}{674}\sin(M\Delta tn)$$

with $\mathrm{n} = 1:\mathrm{N}$ and $\mathrm{N} = \text{number of samples}$

Figure 4 and 5 show the solution for some specific values by using all methods, while figure 4 shows the resulting solution, figure 5 shows the errors taking the Laplace method like a reference.

Box 21

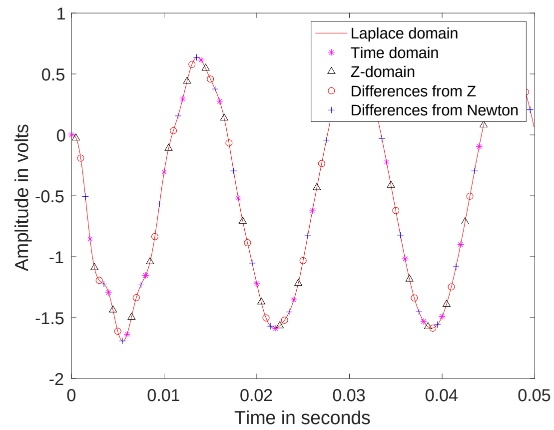

Figure 4
Solution by using A=2, B=3, C=1 and F=1/pi Hz

Box 22

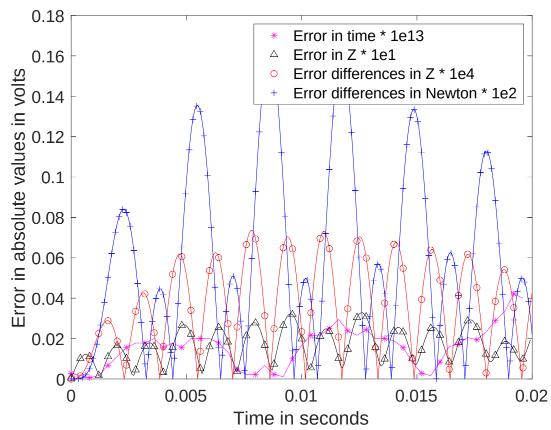

Figure 5
Error taking Laplace like a reference

Table 18 shows the solution for each method, this table only shows the two times steps for which we obtain the best solutions and/or the lowest errors.

Box 23

Table 18
Errors using different values for A, B, C and F
A=-1e4*w^2, B=-2e6*w, C=0 and F=60 Hz.

Kind of error	Time domain	Z-domain	Differences from Z	Differences Newton
MSE_{Error}	4.7435e-26	0.0082716	9.1915e-13	2.2021e-07
	19	9	5	3
	4.7794e-26	0.09869	2.786e-12	1.2037e-06
	12	10	6	4
$AREA_{Error}$	4.6567e-19	0.02863	2.9443e-06	0.0018132
	8	18	6	3
	2.2205e-16	0.034245	3.2246e-06	0.0042225
	9	17	5	4
$\%_{Error}$	1.1081e-17	0.015567	1.1536e-06	0.00071045
	10	12	6	3
	9.9914e-17	0.017007	1.2635e-06	0.0016545
	12	15	5	4

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Third example

The circuit in figure 6 is a classic example of a -cascade that can come from the adjustment of a line, for this case fictitious values were used to show the HOODE methods. The circuit begging in repose conditions, means, the inductors and capacitors are discharged.

Box 24

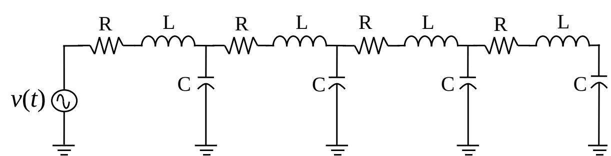

Figure 6

Electrical circuit with R=2 Ω, L=0.5 H and C=0.1 F

The first step is to obtain the transfer function or high-order equation that defines the circuit, this is obtained by analyzing the circuit and is the following:

$$\frac{d^8v}{dt^8}+16\frac{d^7v}{dt^7}+236\frac{d^6v}{dt^6}+1936\frac{d^5v}{dt^5}+12976\frac{d^4v}{dt^4}+56960\frac{d^3v}{dt^3}+176000\frac{d^2v}{dt^2}+320000\frac{dv}{dt}+160000=160000\sin(t)$$

This equation has the following roots:

$r_1=-2+8.1634i, r_2=-2-8.1634i, r_3=-2+6.5533i$
$r_4=-2-6.5533i, r_5=-2+4i, r_6=-2-4i, r_7=-3.26$
and $r_8=-0.73996$; so, with initial conditions equal to zero, the solution is as follows.

Using the Laplace transform

The analytical solution can be obtained by two different methods, the one with indeterminate coefficients or using the Laplace transform, regardless of which one is used, the solution obtained is:

$$\begin{aligned} v(t) &= 2e^{-2t}\left[-0.0008\cos(8.1634t)+0.0015\,\mathrm{sen}(8.1634t)\right] \\ &+2e^{-2t}\left[0.0053\cos(6.5533t)-0.0077\,\mathrm{sen}(6.5533t)\right] \\ &+2e^{-2t}\left[-0.0354\cos(4t)+0.0243\,\mathrm{sen}(4t)\right] \\ &-0.1022e^{-3.26t}+0.7677e^{-0.74t} \\ &+2\left[-0.3019\cos(t)-0.0037\,\mathrm{sen}(t)\right] \end{aligned}$$

Time domain solution

The proposed solution in time domain is as follows,

$$v_T(t)=c_1e^{-r_1t}+c_2e^{-r_2t}+c_3e^{-r_3t}+c_4e^{-r_4t}+c_5e^{-r_5t}+c_6e^{-r_6t}+c_7e^{-r_7t}+c_8e^{-r_8t}+b_1\cos(\omega t)+b_2\sin(\omega t)$$

Time domain particular solution

The particular proposed solution is as follows,
$v_p(t)=b_1\cos(\omega t)+b_2\sin(\omega t)$
which solution yield to
$v_p(t)=-0.60377\cos(\omega t)-0.0074264\sin(\omega t)$

Time domain homogenous solution

Using this solution, we have the total solution as:

$$\begin{aligned} v_T(t) &= c_1e^{-r_1t}+c_2e^{-r_2t}+c_3e^{-r_3t}+c_4e^{-r_4t}+c_5e^{-r_5t} \\ &+c_6e^{-r_6t}+c_7e^{-r_7t}+c_8e^{-r_8t} \\ &-0.60377\cos(\omega t)-0.0074264\sin(\omega t) \end{aligned}$$

by using this proposed solution with $r_1=-2+8.1634i$, $r_2=-2-8.1634i$, $r_3=-2+6.5533i$, $r_4=-2-6.5533i$, $r_5=-2+4i$, $r_6=-2-4i$, $r_7=-3.26$, $r_8=-0.73996$, and the initial conditions, to generate the algebraic system, the final solution is,

$$\begin{aligned} v_T(t) &= (-0.0008-0.0015i)e^{-r_1t}+(-0.0008-0.0015i)e^{-r_2t} \\ &+(0.0052+0.0076i)e^{-r_3t}+(0.0052-0.0076i)^{-r_4t} \\ &+(-0.0353-0.0243i)e^{-r_5t}+(-0.0353+0.0243i)e^{-r_6t} \\ &-0.10217e^{-r_7t}+0.7677e^{-r_8t} \\ &-0.60377\cos(\omega t)-0.0074264\sin(\omega t) \end{aligned}$$

Z-transform solution

The first step to obtain the solution using the z-transform technique is to obtain the transfer function in *z* from the function in *s*, with $h=8x10^{-2}$, from here we obtain:

$$V(z)=\frac{\begin{gathered}7.679\text{x}10^{-10}z^{10}+7.679\text{x}10^{-9}z^9+3.456\text{x}10^{-8}z^8+ \\ 9.215\text{x}10^{-8}z^7+1.613\text{x}10^{-7}z^6+1.935\text{x}10^{-7}z^5+1.613\text{x}10^{-7}z^4+ \\ 9.215\text{x}10^{-8}z^3+3.456\text{x}10^{-8}z^2+7.679\text{x}10^{-9}z+7.679\text{x}10^{-10}\end{gathered}}{\begin{gathered}z^{10}-8.221z^9+30.87z^8-69.8z^7+105.2z^6-110.7z^5+ \\ 82.19z^4-42.59z^3+14.74z^2-3.077z+0.2942\end{gathered}}$$

The previous function is expanded in partial fractions to obtain the residues (*R*), the poles (*P*), and the constant of proportionality (*kp*), from which, we obtain:

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Article

$$R=\begin{bmatrix}-0.024112+0.00029654i\\-0.024112-0.00029654i\\0.061471\\-6.3495e-05-0.00010723i\\-6.3495e-05+0.00010723i\\0.00042055+0.00056017i\\0.00042055-0.00056017i\\-0.0028222-0.0018376i\\-0.0028222+0.0018376i\\-0.0083152\end{bmatrix},\quad P=\begin{bmatrix}0.99681+0.079872i\\0.99681-0.079872i\\0.94251\\0.69674+0.51301i\\0.69674-0.51301i\\0.74883+0.42447i\\0.74883-0.42447i\\0.81208+0.26846i\\0.81208-0.26846i\\0.76928\end{bmatrix},$$

$$kp=2.6101x10^{-9}\quad and\quad h=8x10^{-2}$$

with these data the solution of the differential equation is formed as $v(t)=sum(R.\acute{}\ P^{n})$, where n is the sample number and $(\acute{})$ means a vector operation element by element; the first sample must be corrected with the constant of proportionality with $v(1)=v(1)+kp/h$.

Equation in differences solution from z

When doing the process of having a difference equation starting from the z transform, with $h=2x10^{-2}$,the following expression is reached,

$$\begin{aligned}&v(n-0)-7.6398v(n-1)+25.58v(n-2)-49.03v(n-3)+58.84v(n-4)\\&-45.272v(n-5)+21.809v(n-6)-6.0139v(n-7)+0.72684v(n-8)\\&=8.434x10^{-17}\sin((n-0)h)+6.7472x10^{-16}\sin((n-1)h)\\&+2.3615x10^{-15}\sin((n-2)h)+4.7231x10^{-15}\sin((n-3)h)\\&+5.9038x10^{-15}\sin((n-4)h)+4.7231x10^{-15}\sin((n-5)h)\\&+2.3615x10^{-15}\sin((n-6)h)+6.7472x10^{-16}\sin((n-7)h)\\&+8.434x10^{-17}\sin((n-8)h)\end{aligned}$$

Reducing the right part of the equation with the identity

$$A\sin(n-k)=A\sin(n)\cos(k)-A\cos(n)\sin(k)$$

we obtain,

$$\begin{aligned}&v(n-0)-7.6398v(n-1)+25.58v(n-2)\\&-49.03v(n-3)+58.84v(n-4)-45.272v(n-5)\\&+21.809v(n-6)-6.0139v(n-7)+0.72684v(n-8)\\&=-2.7596x10^{-10}\cos(nh)+3.4422x10^{-9}\sin(nh)\end{aligned}$$

The solution to this equation leads to

$$v(t)=\text{sum}(R.\times P^{n})-0.60376\cos(nh)-0.0074491\sin(nh)$$

with

$$R=\begin{bmatrix}-0.00080109-0.0015378i\\-0.0008011+0.0015378i\\0.0052847+0.0077361i\\0.0052847-0.007736i\\-0.03536-0.024464i\\-0.03536+0.024464i\\0.76748\\-0.10196\end{bmatrix},P=\begin{bmatrix}0.9483+0.15593i\\0.9483-0.15593i\\0.95272+0.12546i\\0.95272-0.12546i\\0.95777+0.076776i\\0.95777-0.076776i\\0.98531\\0.93686\end{bmatrix}$$

$$and\quad h=2x10^{-2}$$

Equation in differences solution from finite Newton differences

The direct implementation of the high-order differential equation in finite differences is unstable, so we proceeded to obtain the difference equation from which the coefficients and roots were obtained, with $h=8x10^{-2}$

$$R=\begin{bmatrix}-0.00087625-0.0016603i\\-0.0008768+0.0016592i\\0.0055132+0.0081145i\\0.0055127-0.0081064i\\-0.036124-0.025104i\\-0.036126+0.025101i\\0.76761\\-0.10348\end{bmatrix},\quad P=\begin{bmatrix}0.9802+0.062997i\\0.9802-0.062997i\\0.98164+0.050687i\\0.98164-0.050687i\\0.98327+0.030904i\\0.98327-0.030904i\\0.99402\\0.97468\end{bmatrix}$$

$$\text{and}\quad h=8x10^{-2}$$

So, the solution is obtained as,

$$v(t)=\text{sum}(R.\times P^{n})+-0.60376\cos(nh)-0.0074491\sin(nh)$$

Analyzing the absolute values of the roots of the difference equation, these are greater than unity and therefore the solution is unstable, for this reason it is not included in the simulation results.

NOTE: For this equation we use only 15-time steps because for short Δt we do not obtain good results. These time steps are listened in table 19.

Box 25

Table 19

Position of each used Δt.

Pos	Δt in sec	Pos	Δt in sec	Pos	Δt in sec
1	2e-05	6	0.0008	11	0.04
2	4e-05	7	0.002	12	0.08
3	8e-05	8	0.004	13	0.2
4	0.0002	9	0.008	14	0.4
5	0.0004	10	0.02	15	0.8

Figure 7 and 8 show the solution by using some data; while figure 7 show the obtained results, figure 8 show the error taking Laplace method like a reference.

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Box 26

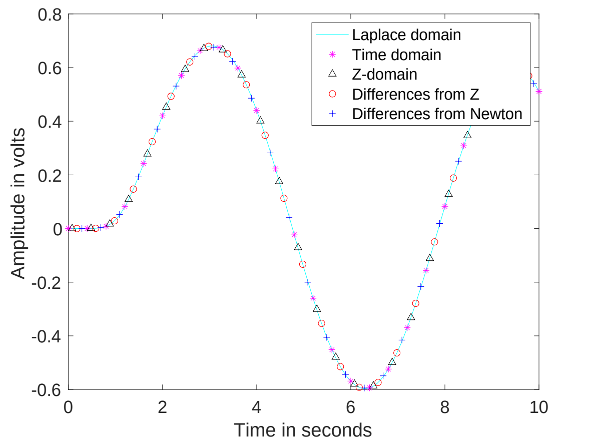

Figure 7

a) Solution by Laplace, z-transform and equation in differences

Box 27

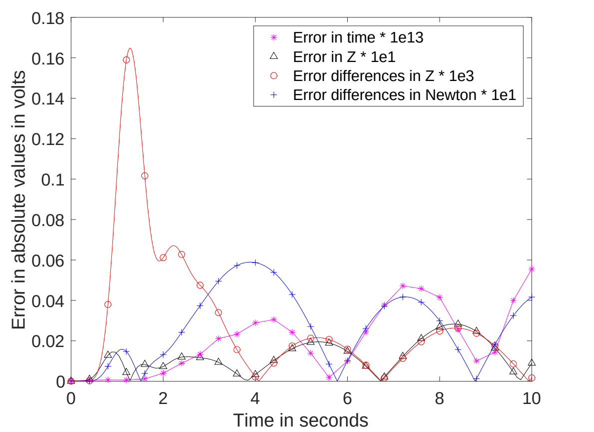

Figure 8

Error taking Laplace like a reference

Table 20 shows the best solution, that means, the time steps for which we obtain the lowest error according with the used error.

Box 28

Table 20

Errors using different values for A, B, C and F

A=-1e4*w^2, B=-2e6*w, C=0 and F=60 Hz.

Kind of error	Time domain	Z-domain	Differences from Z	Differences Newton
MSE_{Error}	6.342e-30	2.0295e-06	2.0983e-09	1.0377e-05
	15	12	10	9
	6.494e-30	2.0813e-05	1.8988e-08	2.5187e-05
	1	11	11	10
$AREA_{Error}$	2.3834e-16	0.0030792	0.00022538	0.0084913
	1	12	10	10
	2.3836e-16	0.019035	0.00066846	0.0090726
	7	13	11	9
$\%_{Error}$	3.8934e-17	0.00044361	3.3041e-05	0.0012562
	15	12	10	10
	5.0282e-17	0.0026695	9.7801e-05	0.0013342
	3	13	11	9

Conclusions

Figure 2 shows the first cycles of the simulation and the error taking as a reference the analytical solution, which was obtained with the Laplace transform. Likewise, Figure 3 shows the long-term simulation, that is, until both the results obtained with the z transform and those of the difference equation based on this transform are apparently stabilized.

At first glance it can be seen how the results obtained starting from the z-transform have an increasing error, the reason is that a sinusoidal function is used as a source and the roots that must be obtained are, however when doing the analysis, they are. They remained in position 4 and 5 of the vector of roots and have an error of, this error, although very small, increases as the number of samples advances. This error of course is strictly numerical but it cannot be removed because it is part of the entire numerical process, that is, if it is arbitrarily removed, the solution is not improved.

Conflict of interest

The authors declare no interest conflict. They have no known competing financial interests or personal relationships that could have appeared to influence the article reported in this article.

Author contribution

Gutiérrez-Robles, José Alberto: Development of the algebra of the methods proposed here.

Galván-Sánchez, Verónica Adriana: Programming the methods presented in the article

Bañuelos-Cabral, Eduardo Salvador: Review, generation and selection of the examples and/or results presented in the article.

De La Cruz-García, Elba Lilia: Review of the content, putting the article into format and translation into English.

Availability of data and materials

All the results that are obtained are in the article and can be accessed freely depending on the journal's policies.

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Abbreviations

HOODE	High-Order Ordinary Differential Equations
ITLS	Invariant in Time Linear Systems
MSE	Mean Square Error
ODE	Ordinary Differential Equation

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